

Altruism in Atomic Congestion Games

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Abstract

This paper studies the effects of introducing altruistic agents into atomic congestion games. Altruistic behavior is modeled by a trade-off between selfish and social objectives. In particular, we assume agents optimize a linear combination of personal delay of a strategy and the resulting social cost. Our model can be embedded in the framework of congestion games with player-specific latency functions. Stable states are the Nash equilibria of these games, and we examine their existence and the convergence of sequential best-response dynamics. Previous work shows that for symmetric singleton games with convex delays Nash equilibria are guaranteed to exist. For concave delay functions we observe that there are games without Nash equilibria and provide a polynomial time algorithm to decide existence for symmetric singleton games with arbitrary delay functions. Our algorithm can be extended to compute best and worst Nash equilibria if they exist. For more general congestion games existence becomes NP-hard to decide, even for symmetric network games with quadratic delay functions. Perhaps surprisingly, if all delay functions are linear, then there is always a Nash equilibrium in any congestion game with altruists and any better-response dynamics converges.

In addition to these results for uncoordinated dynamics, we consider a scenario in which a central altruistic institution can motivate agents to act altruistically. We provide constructive and hardness results for finding the minimum number of altruists to stabilize an optimal congestion profile and more general mechanisms to incentivize agents to adopt favorable behavior.

1 Introduction

Algorithmic game theory has been focused on game-theoretic models for a variety of important applications in the Internet, e.g. selfish routing [5, 8, 28], network creation [4], as well as aspects of e-commerce [18] and social networks [16]. A fundamental assumption in these games, however, is that all agents are *selfish*. Their goals are restricted to optimizing their direct personal benefit, e.g. their personal delay in a routing game. The assumption of selfishness in the preferences of agents is found in the vast majority of present work on economic aspects of the Internet. However, this assumption has been repeatedly questioned by economists and psychologists. In experiments it has been observed that participant behavior can be quite complex and contradictory to selfishness [22, 23]. Various explanations have been given for this phenomenon, e.g. senses of fairness [12], reciprocity among agents [17], or spite and altruism [10, 23].

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Prominent developments in the Internet like Wikipedia, open source software development, or Web 2.0 applications involve or explicitly rely on voluntary participation and contributions towards a joint project without direct personal benefit. These examples display forms of *altruism*, in which agents accept certain personal burdens (e.g. by investing time, attention, and money) to improve a common outcome. While malicious behavior has been considered recently for instance in nonatomic routing [6, 7, 21], virus inoculation [25], or bayesian congestion games [14], a deeper analysis of the effects of altruistic agents on competitive dynamics in algorithmic game theory is still missing.

We consider and analyze a model of altruism inspired by Ledyard [22, p. 154], and recently studied for non-atomic routing games by Chen and Kempe [7]. Each agent i is assumed to be partly selfish and partly altruistic. Her incentive is to optimize a linear combination of personal cost and social cost, given by the sum of cost values of all agents. The strength of altruism of each agent i is captured by her *altruism level* $\beta_i \in [0, 1]$, where $\beta_i = 0$ results in a purely selfish and $\beta_i = 1$ in a purely altruistic agent.

Chen and Kempe [7] proved that in non-atomic routing games Nash equilibria are always guaranteed to exist, even for partially spiteful users, and analyzed the price of anarchy for parallel link networks. In our paper, we conduct the first study of altruistic agents in atomic congestion games, a well-studied model for resource sharing. A standard congestion game is given by a set N of myopic selfish users and a set E of resources. Each resource e has a non-decreasing delay function d_e . Every agent i can pick a strategy S_i from a set of possible strategies $\mathcal{S}_i \subseteq 2^E$, which means she allocates the set S_i of resources (e.g. a path in a network). She then experiences a delay corresponding to the total delay on all resources in S_i , which in turn depends on the number of agents that allocate each resource. Each agent strives to pick a strategy minimizing her experienced delay. A stable state in such a game is a pure Nash equilibrium, in which each agent picks exactly one strategy, and no agent can decrease her delay by unilaterally changing her strategy. The study of congestion games received a lot of attention in recent years, mostly because of the intuitive formulation and their appealing analytical properties. In particular, they always possess a pure Nash equilibrium and every sequential better-response dynamics converges.

As one might expect, the presence of altruists can significantly alter the convergence and existence guarantees of pure Nash equilibria in congestion games. After a formal definition of *congestion games with altruists* in Section 2, we concentrate on pure equilibria and leave a study of mixed Nash equilibria for future work. Our results are as follows.

It is a simple exercise to observe that even in a *singleton game*, in which each strategy consists of a single resource, and for *symmetric* agents, where each agent has the same set of strategies, a Nash equilibrium can be absent. This is the case even for *pure altruists and egoists*, i.e. a population of agents which are either purely altruistic or purely selfish and their $\beta_i \in \{0, 1\}$. However, we show in Section 3 that such games admit a polynomial time algorithm to decide the existence problem. Furthermore, our algorithm can be adapted to compute the Nash equilibrium with best and worst social cost if it exists, for any agent population with a constant number of different altruism levels.

For slightly more general *asymmetric singleton games*, in which strategy spaces of agents differ, we show in Section 3 that deciding the existence of Nash equilibria becomes NP-hard. Nevertheless, for the important subclass of convex delay functions, previous results imply that for any agent population a Nash equilibrium exists and can be obtained in polynomial time. In contrast, we show in Section 4 that convexity of delay functions is not sufficient for more general games. In particular, even for *symmetric network games*, in which strategies represent paths through a network, quadratic delay functions and pure altruists, Nash equilibria can be absent and deciding their existence is

NP-hard. Perhaps surprisingly, if all delay functions are *linear*, then there is a potential function. Thus, for every agent population Nash equilibria exist and better-response dynamics converges.

In addition to these results for uncoordinated dynamics, in Section 5 we consider a slightly more coordinated scenario, in which there is a central institution striving to obtain a good outcome. An obvious way to induce favorable behavior is to convince agents to act altruistically. In this context a natural question is how many altruists are required to stabilize a social optimum. This has been considered under the name “price of optimum” in [20] for Stackelberg routing in nonatomic congestion games. As a Nash equilibrium in atomic games is not necessarily unique, we obtain two measures - an *optimal stability threshold*, which is the minimum number of altruists such that there is *any* optimal Nash equilibrium, and an *optimal anarchy threshold*, which asks for the minimum number of altruists such that *every* Nash equilibrium is optimal. For symmetric singleton games, we adapt our algorithm for computing Nash equilibria to determine both thresholds in polynomial time.

In our model the optimal anarchy threshold might not be well-defined even for singleton games. If all agents are altruists, there are suboptimal local optima in symmetric games with concave delays, or in asymmetric games with linear delays. Hence, even by making all agents altruists, the worst Nash equilibrium sometimes remains suboptimal. In contrast, we adapt the idea of the optimal stability threshold to a very general scenario, in which we can find a stable state with a given, not necessarily optimal, congestion profile. Each agent has a personalized *stability cost* for accepting a strategy under the given congestions. We provide an incentive compatible mechanism to determine an allocation of agents to strategies with minimum total stability cost. Unfortunately, such a general result is restricted to the case of singleton games. Even for symmetric network games on series-parallel graphs, we show that the problem of determining the optimal stability threshold is NP-hard.

2 Model and Initial Results

We consider congestion games with altruists. A *congestion game with altruists* G is given by a set N of n agents and a set E of m resources. Each agent i has a set $\mathcal{S}_i \subseteq 2^E$ of strategies. In a *singleton* congestion game each agent has only singleton strategies $\mathcal{S}_i \subseteq E$. A vector of strategies $S = (S_1, \dots, S_n)$ is called a *state*. For a state we denote by n_e the congestion, i.e. the number of agents using a resource e in their strategy. Each resource e has a *latency* or *delay* function $d_e(n_e)$, and the *delay for an agent* i playing S_i in state S is $d_i(S) = \sum_{e \in S_i} d_e(n_e)$. The *social cost* of a state is the total delay of all agents $c(S) = \sum_{i \in N} \sum_{e \in S_i} d_e(n_e) = \sum_{e \in E} n_e d_e(n_e)$. Each agent i has an *altruism level* of $\beta_i \in [0, 1]$, and her *individual cost* is $c_i(S) = \beta_i c(S) + (1 - \beta_i) d_i(S)$. We call an agent i an *egoist* if $\beta_i = 0$ and a β_i -*altruist* otherwise. A (*pure*) *altruist* has $\beta_i = 1$, a (*pure*) *egoist* has $\beta_i = 0$. A game G with only pure altruists and egoists is a game, in which $\beta_i \in \{0, 1\}$ for all $i \in N$. A game G is said to have β -uniform altruists if $\beta_i = \beta \in [0, 1]$ for every agent $i \in N$. A (*pure*) *Nash equilibrium* is a state S , in which no agent i can unilaterally decrease her individual cost by unilaterally changing her strategy. We exclusively consider pure equilibria in this paper.

If all agents are egoists, the game is a regular congestion game, which has an exact potential function $\Phi(S) = \sum_{e \in E} \sum_{x=1}^{n_e} d_e(x)$ [27]. Thus, existence of Nash equilibria and convergence of iterative better-response dynamics are guaranteed. Obviously, if all agents are altruists, Nash equilibria correspond to local optima of the social cost function c with respect to a local neighborhood consisting of single player strategy changes. Hence, existence and convergence are also guaranteed.

This directly implies the same properties for β -uniform games, in which an exact potential function is $\Phi_\beta(S) = (1 - \beta)\Phi(S) + \beta c(S)$.

In general, however, Nash equilibria might not exist.

Observation 1. *There are symmetric singleton congestion games with only pure altruists and egoists without a Nash equilibrium.*

Example 2. Consider a game with two resources e and f , three egoists and one (pure) altruist. The delay functions are $d_e(x) = d_f(x)$ with $d_e(1) = 4$, $d_e(2) = 8$, $d_e(3) = 9$, and $d_e(4) = 11$. Then, in equilibrium each resource must be allocated by at least one egoist. In case there are two agents on each resource, the social cost is 32. In this case the altruist is motivated to change as the resulting cost is 31. In that case, however, one of the egoists on the resource with congestion 3 has an incentive to change. Thus, no Nash equilibrium will evolve.

Our interest is thus to characterize the games that have Nash equilibria. Towards this end we observe that an altruistic congestion game can be cast as a congestion game with player-specific latency functions [24]. For simplicity consider a game with only pure altruists and egoists. An altruist moves from S_i to S'_i if the decrease in total delay $n_e d_e(n_e)$ on the resources $e \in S_i - S'_i$ she is leaving exceeds the increase on resources $e \in S'_i - S_i$ she is migrating to. Hence, altruists can be seen as myopic selfish agents with $c_i(S) = d'_i(S) = \sum_{e \in S_i} d'_e(n_e)$ with $d'_e(n_e) = n_e d_e(n_e) - (n_e - 1)d_e(n_e - 1)$, for $n_e > 0$. We set $d'_e(0) = 0$. Naturally, a β_i -altruist corresponds to a selfish agent with player-specific function $c_i(S) = (1 - \beta_i)d_i(S) + \beta_i d'_i(S)$. Thus, our games can be embedded into the class of player-specific congestion games. For some classes of these games it is known that Nash equilibria always exist. In particular, non-existence in Example 2 is due to the fact that the individual delay function for the altruist is not monotone. Monotonicity holds, in particular, if delay functions are convex. In this case, it is known that for matroid games, in which the strategy space of each agent is a matroid, existence of a Nash equilibrium is guaranteed [2].

Corollary 3. [2, 24] *For any matroid congestion game with altruists and convex delay functions a Nash equilibrium exists and can be computed in polynomial time.*

3 Singleton Congestion Games

In the previous section we have seen that there are symmetric singleton congestion games with only pure altruists and egoists with and without Nash equilibria. For this class of games we can decide the existence of Nash equilibria in polynomial time. In addition, we can compute a Nash equilibrium with minimum and maximum social cost if they exist.

Theorem 4. *For symmetric singleton games with only pure altruists and egoists there is a polynomial time algorithm to decide if a Nash equilibrium exists and to compute the best and the worst Nash equilibrium.*

Proof. We first tackle the existence problem and present an approach similar to [19] based on dynamic programming. Suppose we are given a game G with the set N_0 of n_0 egoists and the set N_1 of $n_1 = n - n_0$ altruists. For a state S consider the set of resources $E_0 = \bigcup_{i \in N_0} S_i$ on which at least one egoist is located. The maximum delay of any resource on which an egoist is located is denoted $d_0^{max} = \max_{e \in E_0} d_e(n_e)$ and minimum delay of any resource if an additional agent is added $d_0^{min+} = \min_{e \in E} d_e(n_e + 1)$. Similarly, consider the set of resources $E_1 = \bigcup_{i \in N_1} S_i$. The maximum

altruistic delay of any resource, on which an altruist is located, is denoted $d_1^{max} = \max_{e \in E_1} d'_e(n_e)$ and the minimum altruistic delay of any resource $d_1^{min+} = \min_{e \in E} d'_e(n_e + 1)$. A state is a Nash equilibrium if and only if

$$d_0^{max} \leq d_0^{min+} \quad \text{and} \quad d_1^{max} \leq d_1^{min+}. \quad (1)$$

This condition yields a separation property. Consider a Nash equilibrium, in which $n_{E',0}$ egoists and $n_{E',1}$ altruists are located on a subset $E' \subset E$ of resources. The Nash equilibrium respects the inequalities above for certain values $d_{0/1}^{max}$ and $d_{0/1}^{min+}$. Note that it is possible to completely change the assignment of agents in E' . If the new assignment respects the inequalities for the same values, it can be combined with the assignment on $E - E'$ and again a Nash equilibrium evolves.

This property suggests the following approach to search for an equilibrium. Suppose the values for $d_{0/1}^{max}$ and $d_{0/1}^{min+}$ are given. Our algorithm adds resources e one by one and tests the possible numbers of egoists and altruists that can be assigned to e . Suppose we have processed the resources from a subset E' and have found the numbers of altruists and egoists, for which there is an assignment to resources E' such that there is no violation of equations (1) for the given delay values. In this case, we know the feasible numbers of altruists and egoists that are left to be assigned to the remaining resources. Suppose we have marked these combinations of remaining agents in a boolean matrix R of size $(n_0 + 1) \times (n_1 + 1)$. Here $r_{ij} = 1$ if and only if there is a feasible assignment of $n_0 - i$ egoists and $n_1 - j$ altruists to E' . For the new resource e we now test all combinations $(n_{e,0}, n_{e,1})$ of altruists and egoists that can be allocated to e such that the equations (1) remain fulfilled. We then compile a new matrix R' of the feasible combinations of remaining agents for the remaining resources $E - E' - \{e\}$. In particular, for each tuple $(n_{e,0}, n_{e,1})$ and each positive entry r_{ij} of R we check if $i - n_{e,0} \geq 0$ and $j - n_{e,1} \geq 0$. If this holds, we set the entry of R' with index $(i - n_{e,0}, j - n_{e,1})$ to 1. If e is the last resource to be processed, we check if the resulting matrix R' has a positive entry $r'_{0,0} = 1$. In this case a Nash equilibrium exists for the given values of $d_{\{0,1\}}^{max}$ and $d_{\{0,1\}}^{min+}$, otherwise it does not exist. Due to the separation property mentioned above, this approach succeeds to implicitly test all allocations that fulfill equations (1) for the given values. Finally, note that there are only at most $O(n_0^2 n_1^2 m^4)$ possible values for which we must run the algorithm.

The separation property mentioned above also applies to the best or worst Nash equilibrium. In particular, consider the best Nash equilibrium S that respects (1) for some fixed values $d_{\{0,1\}}^{max}$ and $d_{\{0,1\}}^{min+}$. Consider any subset of resources E' with a number $n_{E',0}$ and $n_{E',1}$ of egoists and altruists, respectively. S is the cheapest Nash equilibrium that respects (1) for the given values if and only if the assignment of S in E' is the cheapest assignment with $n_{E',0}$ egoists and $n_{E',1}$ altruists that respects (1) for the values. Thus, we can adjust our approach as follows. For a set E' of processed resources, instead of simply noting in r_{ij} that there is a feasible assignment to E' that leaves i egoists and j altruists, we can remember the social cost of the cheapest of such assignments. Thus, the matrix R is then a matrix of positive entries, for which we use a prohibitively large cost to identify infeasible combinations. When we compile a new matrix R' after testing all feasible assignments to a new resource e , we can denote in each entry the minimum cost that can be obtained for the respective combination. A similar argument works for computing the worst Nash equilibrium. This decides the existence question and finds the cost values of best and worst Nash equilibria. By tracing back the steps of the algorithm we can also discover the strategy choices of agents. \square

Note that the previous proof can be extended to a constant number k of different altruism levels. In this more general scenario we choose the delay parameters for each level of altruists.

For each resource e we then test all possible combinations of agents from the different levels that we can allocate to a resource e and satisfy all bounds. The matrix R changes in dimension to $(n_{\beta_1} + 1) \times \dots \times (n_{\beta_k} + 1)$ to account for all feasible combinations of remaining agents. Finally, we need to test all combinations of delay bounds. However, if k is constant, all these operations can be done in polynomial time.

Corollary 5. *For symmetric singleton games with altruists and a constant number of different altruism levels, there is a polynomial time algorithm to decide if a Nash equilibrium exists and to compute the best and the worst Nash equilibrium.*

As a byproduct, our approach also allows us to compute a social optimum state in polynomial time. We simply assume all agents to be pure altruists and compute the best Nash equilibrium.

Corollary 6. *For symmetric singleton congestion games a social optimum state can be obtained in polynomial time.*

In case of asymmetric games, however, deciding the existence of Nash equilibria becomes significantly harder.

Theorem 7. *It is NP-hard to decide if a singleton congestion game with only pure altruists and egoists has a Nash equilibrium if G is asymmetric and has concave delay functions.*

Proof. We reduce from 3SAT. Given a formula φ , we construct a congestion game G_φ that has a Nash equilibrium if and only if φ is satisfiable. Let x_1, \dots, x_n denote the variables and c_1, \dots, c_m the clauses of a formula φ . Without loss of generality [30], we assume each variable appears at most twice positively and at most twice negatively.

For each variable x_i there is a selfish agent X_i that chooses one of the resources $e_{x_i}^1$, $e_{x_i}^0$, or e_0 . The resources $e_{x_i}^1$ and $e_{x_i}^0$ have the delay function $9x$ and resource e_0 has the delay function $7x + 3$. For each clause c_j , there is a selfish agent C_j who can choose one of the following three resources. For every positive literal x_i in c_j he may choose $e_{x_i}^0$. For every negated literal \bar{x}_i in c_j he may choose $e_{x_i}^1$. Note that there is a stable configuration with no variable agent on e_0 if and only if there is a satisfiable assignment for φ . Additionally, there are three selfish agents u_1 , u_2 , and u_3 who can choose e_1 or e_2 . Each of the resources e_1 and e_2 has delay 4 if used by one agent, delay 8 if used by two agents and delay 9 otherwise. The only pure altruist u_0 chooses between e_1 , e_2 , and e_0 . Note that the altruist chooses e_1 , e_2 if one of the variable agents is on e_0 .

If φ is satisfiable by a bitvector (x_1^*, \dots, x_n^*) , a stable solution for G_φ can be obtained by placing each variable agent x_i on $e_{x_i}^{x_i^*}$. Since (x_1^*, \dots, x_n^*) satisfies φ there is one resource for each clause agent that is not used by a variable agent. Thus, we can place each clause agent on this resource, which he then shares with at most one other clause agent. Let the altruist u_0 use e_0 and u_1 and u_2 choose e_1 and u_3 choose e_2 . It is easy to check that this is a Nash equilibrium.

If φ is unsatisfiable, there is no stable solution. To prove this it suffices to show that one of the variable agents prefers e_0 . In that case the altruist never chooses e_0 and the agent u_0, \dots, u_3 play the sub game of Example 2. For the purpose of contradiction assume that φ is not satisfiable but there is a stable solution in which no variable wants to choose e_0 . This implies that there is no other agent, i.e. a clause agent, on a resource that is used by a variable agent. However, if all clause agents are on a resource without a variable agent we can derive a corresponding bit assignment which, by construction, satisfies φ .

Therefore, G_φ has a stable solution if and only if φ is satisfiable. \square

includes edge e_0 if and only if Φ is not satisfiable. If no variable agent is on e_0 , a Nash equilibrium can be obtained by placing u_1 on the path that begins with (e_1, e_0) . For agent u_0 it is optimal to choose the path (e_8, e_6, e_3) .

However, if at least one variable agent is on the edge e_0 , there is no Nash equilibrium. If the altruist u_0 is on the path (e_8, e_6, e_3) , the best response for u_1 is the path (e_4, e_5, e_6, e_7) . If u_1 is the path (e_4, e_5, e_6, e_7) , the best response for the altruist u_0 is path (e_7, e_4, e_2) . If u_0 is on (e_7, e_4, e_2) , the best response for u_1 is the path that begins with the edge e_{10} . This, finally, is a state in which the best response for u_0 is (e_8, e_6, e_3) . Thus, the constructed network congestion game G_Φ has a Nash equilibrium if and only if the formula Φ is satisfiable.

Now, we turn the asymmetric network congestion game G_Φ into a symmetric congestion game G'_Φ . We add a new source node s , a new target node t and a node s' to the network and connect them to G_Φ as depicted by the dashed edges in Figure 1. Note that M is an integer that is larger than the sum of possible delay values in G_Φ . If all agents play their best responses, then we can observe the following: Each outgoing edge of s' is used by exactly one selfish agent and the altruist chooses a path that begins with the edge (s, s_0) . Every best response path of a selfish agent finishes with the edge (t', t) . Every best response path of the altruist ends with the edge (t_0, t) . Therefore, G'_Φ has a Nash equilibrium if and only if G_Φ has a Nash equilibrium. \square

Perhaps surprisingly, if *every* delay function is linear $d_e(x) = a_e x + b_e$, then an elegant combination of the Rosenthal potential and the social cost function yields a potential for arbitrary β_i -altruists. Hence, existence of Nash equilibria and convergence of sequential better-response dynamics is always guaranteed. The proof is carefully constructed for altruists, as for congestion games with general player-specific linear latency functions a potential does not exist [15]. We only consider delays $d_e(x) = a_e x$ without offset b_e , but as noted earlier, this is not a restriction.

Theorem 9. *For any congestion game with altruists and linear delay functions there is always a Nash equilibrium and sequential better-response dynamics converges.*

Proof. The theorem follows from the existence of a weighted potential Φ that decreases during every improvement step of any agent i with altruism level β_i .

$$\Phi(S) = \sum_{e \in E} \sum_{j=1}^{n_e} a_e j + \sum_{e \in E} a_e n_e^2 - \sum_{i=1}^n \sum_{e \in S_i} \frac{2\beta_i - 1}{\beta_i + 1} a_e$$

Consider a state S and an improving strategy change of an agent i from S_i to S'_i resulting in a strategy profile S' . We show that Φ decreases. For the sake of clarity and brevity we set $\Delta_N = \sum_{e \in S_i \setminus S'_i} a_e n_e - \sum_{e \in S'_i \setminus S_i} a_e (n_e + 1)$ and $\Delta_C = \sum_{e \in S_i \setminus S'_i} (2a_e n_e - a_e) - \sum_{e \in S'_i \setminus S_i} (2a_e n_e + a_e)$. Note that an improving strategy change requires $(1 - \beta) \Delta_N + \beta \Delta_C > 0$.

$$\begin{aligned} \Phi(S) - \Phi(S') &= \Delta_N + \Delta_C - \sum_{e \in S_i \setminus S'_i} \frac{2\beta_i - 1}{\beta_i + 1} a_e + \sum_{e \in S'_i \setminus S_i} \frac{2\beta_i - 1}{\beta_i + 1} a_e \\ &= \left(1 - \frac{2(2\beta_i - 1)}{1 + \beta_i}\right) \Delta_N + \Delta_C + \frac{2(2\beta_i - 1)}{1 + \beta_i} \Delta_N - \sum_{e \in S \setminus S'_i} \frac{2\beta_i - 1}{\beta_i + 1} a_e + \sum_{e \in S'_i \setminus S} \frac{2\beta_i - 1}{\beta_i + 1} a_e \\ &= \left(1 - \frac{2(2\beta_i - 1)}{1 + \beta_i}\right) \Delta_N + \Delta_C + \frac{(2\beta_i - 1)}{1 + \beta_i} \Delta_C \end{aligned}$$

$$= \frac{3 - 3\beta_i}{1 + \beta_i} \Delta_N + \frac{3\beta_i}{1 + \beta_i} \Delta_C = \frac{3}{1 + \beta_i} ((1 - \beta_i) \Delta_N + \beta_i \Delta_C) > 0$$

□

Unfortunately, it follows directly from previous work [11] that the number of iterations to reach a Nash equilibrium can be exponential, and the problem of computing a Nash equilibrium is PLS-hard. For regular congestion games with matroid strategy spaces [1] Nash dynamics converge in polynomial time. It is an interesting open problem if a similar result holds here.

5 Stabilization Methods

This section treats a model in which an institution can convince selfish agents to act as altruists. For simplicity of presentation we first restrict to games with only pure altruists and egoists. A natural question for such an institution to consider is how many altruists are required to guarantee that there is a Nash equilibrium with a certain cost, e.g. a Nash equilibrium as cheap as a social optimum state. A similar question has been considered for Stackelberg routing in the Wardrop model [20, 29]. We term this number the *optimal stability threshold*. In a more pessimistic direction it is of interest to determine the minimum number of altruists needed to guarantee that the worst-case Nash equilibrium is optimal. We term this number the *optimal anarchy threshold*. Let us denote by n_1^+ and n_1^- the optimal stability and anarchy threshold, respectively. As a consequence from Theorem 4 we can compute both numbers for symmetric singleton congestion games in polynomial time. For each number of altruists we check if the best and/or worst Nash equilibrium is as cheap as the social optimum.

Corollary 10. *For symmetric singleton congestion games with only pure altruists and egoists there is a polynomial time algorithm to compute n_1^+ and n_1^- .*

Note that the optimal anarchy threshold is not well-defined, because the worst Nash equilibrium might always be suboptimal, even for a population of altruists only. In case of symmetric singleton games and convex delay functions, an easy exchange argument serves to show that in this case any local optimum is also a global optimum. However, for concave delay functions or asymmetric singleton games, a local optimum might still be globally suboptimal.¹ Note that for symmetric games, our algorithm is able to detect the cases in which suboptimal local optima exist. In the asymmetric case, however, a similar approach fails, because of the NP-hardness of determining existence of a Nash equilibrium. Thus, in the following we concentrate on the optimal stability threshold.

In asymmetric games, it is also required to determine the identity of agents, so here we strive to find a set (denoted N_e^+) of minimum cardinality. For an optimal set of congestion values $n_E^* = (n_e^*)_{e \in E}$ we can determine $N_1^+(n_E^*)$ such that there is a Nash equilibrium of the game with congestion values n_e^* for all $e \in E$.

Theorem 11. *For singleton games with only pure altruists and egoists and a social optimal congestion vector n_E^* there is a polynomial time algorithm to compute $N_1^+(n_E^*)$.*

¹Consider a symmetric game with two resources, $d_1(1) = 16$, $d_1(2) = 32$, $d_1(3) = 36$, and $d_2(x) = 45$. If all agents allocate resource 1, we get a Nash equilibrium of cost 108. In the optimum two agents allocate resource 2 resulting in a cost of 106. Now consider an asymmetric game with three resources and delay functions $d_1(x) = d_2(x) = 8x$, and $d_3(x) = 4x$. Agent 1 can use resources 1 and 2, agents 2 and 3 can use resources 2 and 3. The state (2, 3, 3) is a Nash equilibrium of cost 32, while the social optimum is a state (1, 2, 3) of cost 20.

Proof. Suppose we are given a congestion vector $(n_e^*)_{e \in E}$ that results in minimum social cost. We now construct a weighted bipartite graph as follows. One partition is the set of agent N . In the other partition we introduce for each resource e a number of n_e^* vertices. If $e \in \mathcal{S}_i$ we connect agent i to all vertices that were introduced due to e . If e represents a best-response for i , then we assign a weight of 0 to all corresponding edges between i and the vertices of e . To all other edges we assign a weight of 1. Note that any feasible allocation of agents to strategies that generates the congestion vector n_e^* is represented by a perfect matching. Due to social optimality an altruist can be matched with any strategy, while an egoist must be matched to a best response. If we match an agent to a strategy, which is not a best-response, it thus has to become an altruist and a weight of one is counted towards the weight of the matching. By computing a minimum weight perfect matching [9], we can identify a minimal set $N_1^+(n_E^*)$ of altruists required. \square

Observe that by creating the edges of cost 1 only to strategies which represent best-responses with respect to the altruistic delay d' , we can compute $N_1^+(n_E)$ for arbitrary congestion vectors n_E . In this case, the set might be empty, if e.g. the congestion vector corresponds to a very expensive state and can never be generated by a Nash equilibrium for any distribution of altruists. This case, however, can be recognized by the absence of a perfect matching in the bipartite graph.

This approach turns out to be applicable to an even more general natural scenario. Suppose each agent i has a *stability cost* c_{ie} for each strategy $e \in \mathcal{S}_i$. This cost yields the disutility for being forced to play a certain strategy given a congestion vector n_E . In this scenario we slightly change $N_1^*(n_E)$ to the set agents of minimal stability cost. Still, we can compute this set by a minimum weight perfect matching if we set the weights to c_{ie} for all edges connecting i to vertices of e . The stability cost allows for general preferences exceeding categories like altruists and egoists.

Corollary 12. *For singleton games and a congestion vector n_E there is a polynomial time algorithm to compute $N_1^+(n_E)$ with minimal stability cost.*

The underlying problem can be seen as a slot allocation to agents. As the computed allocation has minimal stability cost, it is possible to turn the algorithm into a truthful mechanism using VCG payments (see e.g. [26, chapter 9]). Our final mechanism (1) learns the stability costs from each agent, (2) determines the allocation, and (3) pays appropriate amounts to agents for truthful revelation of cost values and adaptation of allocated strategies. In addition, it can be verified that all computations needed require only polynomial time.

Corollary 13. *For singleton games and a congestion vector n_E there is a truthful VCG-mechanism to compute $N_1^+(n_E)$ in polynomial time.*

These general results are restricted to the case of singleton games. For more general games we show that it is NP-hard to decide if there is a Nash equilibrium as cheap as the social optimum. Our next theorem establishes this even for symmetric network congestion games with linear delays, in which an arbitrary Nash equilibrium and a social optimum state can be computed in polynomial time [11]. Furthermore, the result requires only a series-parallel network. Thus, even in this restricted case it is NP-hard to decide if the number n_1^+ of pure altruists required is 0 or 1, or equivalently if $N_1^+(n_E^*)$ is empty or not.

Theorem 14. *For symmetric network congestion games with 3 agents, linear delay functions on series-parallel graphs and optimal congestions n_E^* it is NP-hard to decide if there is a Nash equilibrium with congestions n_E^* .*

Proof. We reduce from PARTITION. Let an instance be given by positive integers a_1, \dots, a_k and $a = \sum_{i=1}^k a_i$, where a is an even number. Create a network with two nodes and two parallel edges e_1 and e_2 for each integer a_i . The delay $d_{e_1}(x) = 2a_i x$, and $d_{e_2}(x) = a_i x$. All these networks are concatenated sequentially. We denote the first node of this path gadget by u and the last by v . In addition, we add one edge $f = (u, v)$ with delay $d_f(x) = \frac{3}{4}ax$. Finally, the game has three egoists, which need to allocate a path from u to v .

The unique social optimum is to let one agent use f and the other two agents use two edge-disjoint paths through the path gadget. This yields an optimal social cost of $\frac{15}{4}a$. However, for a Nash equilibrium each path through the gadget must not have more delay than $\frac{3}{2}a$. If the instance of PARTITION is solvable, then the elements assigned to a partition represent the edges of type e_1 that an agent allocates in Nash equilibrium. Otherwise, if the instance is not solvable, there is no possibility to partition the path gadget into two edge-disjoint paths of latency at most $\frac{3}{2}a$.

The reduction works for a small constant number of agents but only shows weak NP-hardness. If the number of agents is variable, it is possible to show strong NP-hardness with a similar reduction from 3-PARTITION. \square

We remark that the previous theorem contrasts the continuous non-atomic case, in which a minimal fraction of altruistic demand stabilizing an optimum solution can be computed in any symmetric network congestion game [20].

6 Conclusions

In this paper, we have initiated the study of altruists in atomic congestion games. Our model is similar to the one presented by Chen and Kempe [7] for nonatomic routing games, however, we observe quite different properties. In the nonatomic case, existence of Nash equilibria for any population of agents is always guaranteed, even if agents are partially spiteful. In contrast, our study answers fundamental questions for existence and convergence in atomic games. For the case of linear latencies, an elegant combination of social cost and the Rosenthal potential proves guaranteed existence and convergence. The next step is to consider the price of anarchy and the relations to results on Stackelberg games [13]. An altruistic variant of the price of malice [3, 25] measuring the influence of altruists on the worst-case Nash equilibrium can be interesting to consider. Finally, a characterization of games for which Nash equilibria exist and best-response dynamics converge (in polynomial time) is an important open problem.

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